Pergamon

J. Appl. Maths Mechs, Vol. 58, No. 4, pp.645–655, 1994

0021–8928/94 $24.00+0.00

0021–8928(94)00090-5

THE ASYMPTOTIC FEATURES OF THE UNSTEADY EXPANSION OF AN IDEAL GAS INTO A VACUUM†

A. N. KRAIKO

Moscow

(*Received* 18 *February* 1993)

Further developing the results obtained in [1], where the asymptotic form of the one-dimensional expansion of an ideal gas with adiabatic index $\kappa>1$ was investigated, the features of the unsteady expansion of an ideal gas (non-viscous and non-heat-conducting) into a vacuum are investigated. If t is the time and x is a coordinate measured from the plane, axis or centre of symmetry, the formulas obtained in [1], which take into account the effect of the vanishing pressure on the inertial expansion of the gas, hold in the region of the plane xt, elongated in the direction of the t axis. The approach used below is free from this limitation, and the relations obtained hold everywhere far from the origin of coordinates. In addition to this, asymptotic formulae are obtained which describe the spherically symmetric inertial expansion of a gravitating gas, and an asymptotic analysis is carried out for an ideal gas with $\kappa=1$. The corrections for gravitation, like the formulae for the inertial expansion of a gas into a vacuum, are independent of its thermodynamic properties. The results obtained hold for times t for which, as a result of the expansion, the volume occupied by the gas considerably exceeds its initial value.

Like [1], the present investigation is related to the work described in [2–9] on steady hypersonic flows, where the inertial expansion of a gas was analysed [2–7] and corrections were obtained due to the vanishing pressure [8, 9].

1. Suppose $\mathbf{u}$ is the velocity, p and ρ are the pressure and the density, and T, h and s are the temperature, the specific enthalpy and the specific entropy of the gas, which are known functions of p and ρ. Then the flow of an ideal gas is described by the equations [10]

$$\frac{d\mathbf{u}}{dt}+\frac{1}{\rho}\nabla p=0,\quad \frac{d\rho}{dt}+\rho\nabla\mathbf{u}=0,\quad \frac{ds}{dt}=0 \tag{1.1}$$

$$\frac{d\mathbf{u}}{dt}+\nabla h-T\nabla s=0\quad \left(\frac{d}{dt}=\frac{\partial}{\partial t}+\mathbf{u}\nabla\right)$$

the fourth of which is a consequence of the first and the equation $Tds=dh-\rho^{-1}dp$.

When $t=0$ the gas occupies the volume Ω_0 with surface $\partial\Omega_0$, which separates it from the surrounding empty space. We will take the length L_0, characterizing the dimensions of Ω_0 as the spatial scale. When Ω_0 has a plane or axis of symmetry, L_0, is the distance from them to $\partial\Omega_0$. The surface $\partial\Omega$ can be purely geometrical or a shell which disappears at the instant $t=0$. In any case, the initial distributions of the parameters, to which we will ascribe the subscript "zero", are arbitrary for zero total momentum of the gas (due to the choice of the system of

†*Prikl. Mat. Mekh.* Vol. 58, No. 4, pp. 70–80, 1994.

coordinates). It is assumed that there are no external forces when $t>0$ and that the origin of coordinates coincides with the centre of mass of the gas.

For arbitrary u_0 the expanding gas may decompose into several unconnected "clouds". The less "exotic" initial conditions (for example, $p_0 \equiv \text{const}$, $\mathbf{u}_0 \equiv 0$) ensure that the expanding gas evolves as a connected whole. In general, the rate of discharge into the vacuum is different at different points of $\partial\Omega_L$. The maximum value u^m of its component normal to $\partial\Omega_L$ will be taken as the scale of u, L_0/u^m will be taken as the scale of t, and the constant c_υ with the dimensions of specific heat capacity will be taken as the scale of s. Then, the last equation of (1.1) becomes

$$\frac{d\mathbf{u}}{dt}+\nabla\frac{h}{u^{m2}}-\frac{c_\upsilon T}{u^{m2}}\nabla s=0 \tag{1.2}$$

The initial fields of the parameters may contain jumps or lead to the formation of such. If the jumps arise due to energy release when $t<0$, we will take as the origin of the coordinate of t the instant it arrives at the furthest point of $\partial\Omega_L$. In any case, however, the rarefaction waves due to the expansion of the gas into the vacuum will, finally, lead to the disappearance of the jumps and to the termination of the increase in s. Hence, due to the increase in the dimensions L of the region Ω, which, for the chosen scale of $\mathbf{u}$, increases, like t, for large t we will have $\nabla s=O(1/t)$. For the same reason the change in $\mathbf{u}$ (within Ω when $t=\text{const}$) and the velocity itself are quantities of the order of unity and, consequently, $(\mathbf{u}\nabla)\mathbf{u}=O(1/t)$. On the other hand, after the jumps disappear, h/u^{m2} will approach zero everywhere in Ω as t increases and $\nabla(h/u^{m2})=o(1/t)$. Due to the reduction in T the same holds for the third term in (1.2). Thus, in (1.2) the second and third terms are of the order of $o(1/t)$, while $(\mathbf{u}\nabla)\mathbf{u}=O(1/t)$. Hence $\partial\mathbf{u}/\partial t=O(1/t)$, and for large values of t (1.2) reduces in the "principal orders" to the equation of the inertial expansion

$$d\mathbf{u}/dt=0 \tag{1.3}$$

According to (1.3) the velocity $\mathbf{u}$ does not change along the trajectories of the gas particles. For even larger t the trajectories are rays: $r=ut$ with radius-vector $\mathbf{r}$ drawn from the origin of coordinates. Hence, by obtaining $\nabla\mathbf{u}=3/t$, after substitution into the second equation of (1.1), we obtain that along the trajectories of the particles

$$\rho=\rho_i(t_i/t)^j \tag{1.4}$$

In (1.4) $j=3$, and t_i is such that when $t\geqslant t_i$ in Ω Eq. (1.2) reduces in "principal orders" to (1.3), while in the equations of the trajectories: $r=ut$ the neglected terms are small compared with $\mathbf{r}$. the density ρ_i in (1.4) varies from particle to particle. According to the third equation of (1.1) along the trajectory when $t\geqslant t_i$

$$s(p,\rho)=s(p_i,\rho_i) \tag{1.5}$$

For an ideal gas we have from (1.4) and (1.5)

$$p=p_i(t_i/t)^{j\kappa} \tag{1.6}$$

Suppose that when there is a centre, axis or plane of symmetry, u is the x-component of $\mathbf{u}$ (in the spherically symmetric case the other components of $\mathbf{u}$ are zero). Then, when $t\geqslant t_i$

$$u=x/t \tag{1.7}$$

and, by virtue of the second equation of (1.1), written taking the corresponding symmetry into account, in (1.4) and (1.6) $j=1+\nu$ with $\nu=0$, 1 and 2 in the plane, cylindrical and spherical cases. Since when $\nu = 0$ and 1 the gas, due to symmetry, does not expand in all directions, in these cases as t increases ρ falls more slowly than for a spherical and arbitrary spatial expansion.

When there is initial swirling when $\nu = 1$ the angular momentum $\gamma = x\upsilon$ is conserved in the particle, where υ is the circular component of $\mathbf{u}$, i.e. along the trajectories of the particles $\upsilon = \gamma/x = \gamma(ut)$, and as t increases this component falls as $1/t$, and the term υ^2/x which occurs in the equation for $\mathbf{u}$, decreases as $1/t^3$. Consequently, here also for sufficiently large $t \geq t_i$ in the "principal orders" u is described by the equation of inertial expansion (1.3) with $\mathbf{u}$ replaced by u.

2. In the one-dimensional case when $t \gg t_i$ the volume Ω_i is represented by a point, and in view of the fact that there is no characteristic linear dimension it is natural to attempt to construct a self-similar solution of the equations defining u and ρ. For such t, from the mass m and the energy E of the expanding gas, defining the parameters of the problem which affect u and ρ, we form the quantity $c = \sqrt{(E/m)}$ with the dimensions of velocity. Then, using the well-known ideas of the theory of dimensions [11], we can write u and ρ in the form

$$u(x,t) = cU(\xi), \quad \rho(x,t) = \frac{m}{(ct)^{1+\nu}} R(\xi), \quad \xi = \frac{x}{ct} \tag{2.1}$$

Substituting (2.1) into (1.3) and into the second equation of (1.1) we arrive at the following system (the prime denotes a derivative with respect to ξ)

$$(U - \xi)U' = 0, \quad (U - \xi)R' = (1 + \nu - U' - \nu U/\xi)R \tag{2.2}$$

The first equation has two solutions: $U' = 0$ and

$$U = \xi \tag{2.3}$$

If $U' = 0$, we have $U(\xi) \equiv \text{const}$. By virtue of the condition of symmetry $U(0) = 0$, whence we have $U(\xi) \equiv 0$, which does not describe expansion into a vacuum. The situation is also not saved by assuming a vacuum in the neighbourhood of $\xi = 0$, since then the unique acceptable solution of the second equation from (2.2) is $R(\xi) = 0$. There remains the solution (2.3), which is identical with (1.7). Here, however, in the second equation of (2.2) the factors in front of R' and R vanish. Hence, $R(\xi)$ is an arbitrary function which satisfies the integral laws of conservation of mass and energy. If ξ^m is the maximum value of ξ, which defines the motion of the boundary $\partial\Omega$, then, taking (2.1) into account and the fact that $c^2 = E/m$, they have the form

$$\int_0^{\xi^m} \xi^\nu R(\xi) d\xi = 1, \quad \int_0^{\xi^m} \xi^{2+\nu} R(\xi) = 2 \tag{2.4}$$

In the approximation considered, Eq. (1.7) for u does not contain p, and when $\nu = 1$ also $\gamma = x\upsilon$, which would also lead to a self-similar solution of (2.1). Taking p into account the three equations of one-dimensional flow would have to be solved together, taking into account the specific form of $s(p, \rho)$. Thus, for an ideal gas $s = s_0 + c_\upsilon \ln[p/(k\rho^\kappa)]$ with dimensional constants s_0, c_υ and k. When $\kappa \neq 1$, from k, m, E, x and t we can set up a dimensionless variable which differs from ξ, which makes the problem a non-self-similar one.

In approximation (1.7) not only is a self-similar solution obtained for u and ρ, but the definition of the other variables is also simplified. The entropy, and for $\nu = 1$, γ also, according to the accurate equations

$$ds/dt = 0, \quad d\gamma/dt = 0 \tag{2.5}$$

are conserved along the trajectories of the particles. In this approximation the trajectories are straight lines $\xi = \text{const}$ and we have from (2.5)

$$s[p(x,t), \frac{m}{(ct)^{1+\nu}} R(\xi)] = c_\upsilon S(\xi), \quad \upsilon(x,t) = \frac{\gamma(\xi)}{ct\xi} \tag{2.6}$$

The functions $S(\xi)$ and $\gamma(\xi)$ that occur here, like $R(\xi)$, are not found from the self-similar equations, where the second equation of (2.6), unlike the first, within the framework of the dimensional analysis, generally cannot be obtained from (2.5). In fact, substituting $s(x, t) = c_\upsilon S(\xi)$ and $\gamma(x, t) = c^2 t\Gamma(\xi)$ into (2.5), we obtain the equations

$$(U - \xi)S' = 0, \quad (U - \xi)\Gamma' + \Gamma = 0$$

Hence it follows that for $U = \xi$ the function $S(\xi)$ is arbitrary, while $\Gamma(\xi) \equiv 0$. Although the equation $\Gamma(\xi) \equiv 0$ contradicts the integral $\gamma = \gamma(\xi)$, which follows from (2.5), with an arbitrary function $\gamma(\xi)$, this contradiction, however, is apparent. When $t \gg t_i$, when $x_i \leqslant L_i \ll L \sim ct$, we have $\Gamma(\xi) = \gamma/(c^2 t) \sim x_i/L \approx 0$, because in the scale $L \sim ct$ the initial dimensions of L_i become a point and it makes no sense to speak of swirling that differs from zero. If we take cL_i as the scale of γ, then putting $\gamma(x, t) = cL_i\Gamma(\xi)$, to determine the function $\Gamma(\xi)$ of the order of unity we arrive at the same equation as for S, and at the second equality of (2.6).

Similar features occur if we determine p from the first equation of (2.6). For an ideal gas this gives

$$p(x,t) = \frac{m^\kappa}{(ct)^{(1+\nu)\kappa}} P(\xi), \quad P(\xi) = k(\xi)[R(\xi)]^\kappa \tag{2.7}$$

with the dimensional entropy function $k(\xi) = p_i \rho_i^{-\kappa}$. For free expansion of a uniform gas initially at rest $k(\xi) = \text{const}$. If we forget the fact that a gas expanding "from a point" when $t \gg t_i$ "remembers" the initial entropy distribution, which affects the self-similar solution of (2.6) or (2.7) in terms of the scale for p, then from an analysis of the dimensions, instead of (2.6) and (2.7), we obtain

$$p(x,t) = \frac{mc^2}{(ct)^{(1+\nu)}} P(\xi) \tag{2.8}$$

Substituting (2.1) and (2.8) into the first equation of (2.5) rewritten in the form $dp/dt = a^2 d\rho/dt$, where a is the velocity of sound, we obtain for an ideal gas

$$(U - \xi)RP' - \kappa(U - \xi)PR' + (1 + \nu)(\kappa - 1)RP = 0$$

Hence, by virtue of (2.3) we have $P(\xi) \equiv 0$. Since, by (2.1) and (2.8)

$$\frac{p}{\rho^\kappa} = \frac{c^2 (ct)^{(1+\nu)(\kappa-1)} P(\xi)}{m^{\kappa-1}[R(\xi)]^\kappa}$$

then $P(\xi) \equiv 0$ provides the unique possibility of conserving $p\rho^{-\kappa}$ along the trajectories of the particles. As in the situation when $\Gamma(\xi) \equiv 0$ the solution (2.8) with $P(\xi) \equiv 0$, obtained by standard dimensional analysis, is the result of the unfortunate scaling of p. Its natural scale, leading to non-trivial solutions of (2.6) and (2.7) is related to the entropy "remembered" by the gas.

The presence in the self-similar solution (2.1), (2.3) and (2.6) of the arbitrary functions $R(\xi)$, $S(\xi)$, $k(\xi)$ and $\gamma(\xi)$ reflects the prehistory of the expansion of the gas when $t < t_i$. It is then, in accordance with the initial conditions when $t = 0$ in $\Omega_0 \ll \Omega_i$ that the processes described by the complete equations (1.1) and the relations on the discontinuities, form density, entropy and swirling fields defining these functions before $t = t_i$. The same initial conditions and processes also define the value of ξ^m, and, in the case when a vacuum is formed in the neighbourhood of the t axis, the minimum non-zero value of ξ.

The effect of the initial non-self-similar conditions on the self-similar solution described above recalls problems with "incomplete self-similarity" [12], but which nevertheless differ in principle from the examples considered in [12]. The structure of the required solution above, as in [11], is found by analysing the dimensions and by the simplest integrals of the flow equations.

3. In the asymptotic formulas of the first approximation from Sections 1 and 2, the expressions for $\mathbf{u}$, u, ρ and υ are universal—they do not depend on the equation of state. Formulae (1.6), (2.6) and (2.7) for p are not universal in any approximation. Since to refine the formulae obtained we must take p into account in the equation of motion, the result of such a refinement depends on the thermodynamics of the gas. By confining ourselves to the one-dimensional case, we initially obtain the required asymptotic expressions for an ideal gas with $\kappa>1$. We start by refining (2.3) for the velocity. Then we obtain the deviation of the trajectories from straight lines: $x=\xi ct$ with $U=\xi=\text{const}$ and we investigate how this affects the asymptotic formulae for ρ and p.

There are two reasons for the errors of formula (2.3): $U=\xi$. First, the distribution of $U\equiv u/c$ when $t=t_i$, even for particles which move with constant velocity, deviates from $U=\xi$, i.e. from the self-similar solution (2.3) on $\delta(\xi)$, although in the inertial expansion mode $\delta\to 0$ as $t_i\to\infty$. When $\delta(\xi)\neq 0$ and for finite t_i, even if Eq. (1.3) is accurate for $t\geqslant t_i$ the trajectories of the particles will not be rays: $x=\xi ct$, emerging from the origin of coordinates, but straight lines

$$x=x_i+(\xi_i+\delta_i)c(t-t_i) \tag{3.1}$$

which intersect the t axis at different points. In (3.1) and henceforth $\delta_i=\delta(\xi_i)$ and by definition $\delta(0)=0$. Along the trajectories, by virtue of (1.3), as before $U=\text{const}$, but now, unlike (2.3)

$$U(x,t)\equiv U(\xi_i)=\xi_i+\delta_i=\frac{x-x_i}{c(t-t_i)}\neq\frac{x}{ct}=\xi \tag{3.2}$$

Second, in (1.3) we have omitted terms proportional to $\partial p/\partial x$, and when $\nu=1$ terms proportional to υ^2 also. After a long time interval they may change the form of the trajectories. In the approximation considered, both effects can be taken into account separately and then summed. The effect of $\delta_i\not\equiv 0$ is expressed by (3.1) and (3.2). We will now take into account the effect of $\partial p/\partial x$. The exact "predecessor" of Eq. (1.3) has the form

$$\frac{du}{dt}+\frac{1}{\rho}\frac{\partial p}{\partial x}-\nu(2-\nu)\frac{\upsilon^2}{x}=0 \tag{3.3}$$

We find the second term in (3.3) from the "first approximation"

$$\frac{1}{\rho}\frac{\partial p}{\partial x}=-\frac{F(\xi)}{(ct)^{n+1}}\equiv-\frac{m^{\kappa-1}P'(\xi)}{(ct)^{n+1}R(\xi)},\quad n=(1+\nu)(\kappa-1) \tag{3.4}$$

and we omit the third. Since $\upsilon^2/x=\gamma^2/x^3=\gamma^2(\xi ct)^{-3}$, by virtue of this for $\nu=1$ the last formulae hold either for $\gamma\equiv 0$ and any $\kappa>1$, or for $\gamma\not\equiv 0$ but $\kappa<2$, when the third term in (3.3) is much less than the second.

Putting $u=c\xi+c\Delta U(x,\ t)$, after substituting (3.4) and this expression into (3.3) and neglecting terms that are quadratic in ΔU, we obtain the equation

$$\frac{\partial\Delta U}{\partial t}+c\xi\frac{\partial\Delta U}{\partial x}+\frac{1}{t}\Delta U=\frac{F(\xi)}{(ct)^{n+1}} \tag{3.5}$$

Its solution, by what was said above, must satisfy the conditions

$$\Delta U(0,\ t)=\Delta U(x,\ t_i)=0 \tag{3.6}$$

Applying separation of variables to (3.5) and (3.6) and taking into account the fact that, by virtue of symmetry $F(0) \sim P'(0) = 0$, we obtain after simple algebra

$$\Delta U = \tau(t)\varphi(\xi), \quad \tau(t) = [\Phi(ct) - \Phi(ct_i)]/(ct) \tag{3.7}$$

$$\varphi(\xi) = \frac{F(\xi)}{c^2}, \quad \Phi(z) = \begin{cases} z^{1-n}/(1-n) & \text{for} \quad n \neq 1 \\ \ln z & \text{for} \quad n = 1 \end{cases}$$

In the "first approximation" along the trajectories $\xi = \text{const}$. Due to the correction (3.7) to the velocity this property is now not satisfied even when $\delta_i \equiv 0$ and $\xi = \xi_i + \Delta\xi$, where by definition $\Delta\xi(\xi_i, t_i) = 0$. To obtain $\Delta\xi$ we will use the fact that along the trajectories $dx/dt = u = c(\xi + \Delta U)$ with ΔU from (3.7), and the fact that $\xi = x/(ct)$. Differentiating the last equation and omitting the product $\varphi'\Delta\xi$, which is small in this approximation, we obtain the equation

$$\frac{d\Delta\xi}{d(ct)} = \frac{\varphi_i}{(ct)^2}[\Phi(ct) - \Phi(ct_i)]$$

Integrating this with the condition $\Delta\xi(\xi_i, t_i) = 0$ we obtain

$$\Delta\xi = \varphi_i[\Phi(ct_i)\left(\frac{1}{ct} - \frac{1}{ct_i}\right) + \chi(ct) - \chi(ct_i)] \tag{3.8}$$

$$\chi(z) = \begin{cases} 1/[n(n-1)z^n] & \text{when} \quad n \neq 1 \\ -(1+\ln z)/z & \text{when} \quad n = 1 \end{cases}$$

Substituting (3.7) and (3.8) into the equation of the trajectory

$$dx/dt = U(\xi_i, t) = \xi + \Delta U = \xi_i + \Delta\xi + \Delta U$$

dropping the quadratic terms and carrying out the integration, we obtain

$$x = x_i + [\xi_i + \frac{\varphi_i}{n(ct_i)^n}]c(t - t_i) + \varphi_i[\Lambda(ct) - \Lambda(ct_i)] \tag{3.9}$$

$$\Lambda(z) = \begin{cases} z^{1-n}/[n(n-1)] & \text{when} \quad n \neq 1 \\ -\ln z & \text{when} \quad n = 1 \end{cases}$$

Combining (3.2) with (3.7) and (3.1) with (3.9) and neglecting small higher-order terms we obtain the following asymptotic formulae of the "second approximation"

$$\begin{aligned} U(\xi_i, t) &= \xi_i + \delta_i + \frac{\varphi_i}{n}\left[\frac{1}{(ct_i)^n} - \frac{1}{(ct)^n}\right] \\ x(\xi_i, t) &= x_i + \left[\xi_i + \delta_i + \frac{\varphi_i}{n(ct_i)^n}\right]c(t - t_i) + \varphi_i[\Lambda(ct) - \Lambda(ct_i)] \simeq \\ &\simeq x_i\alpha_i\frac{t}{t_i}\left(1 - \frac{\delta_i t_i}{\xi_i t} + \frac{\varphi_i}{\xi_i}\Psi\right) \\ \alpha_i &= 1 + \frac{\delta_i}{\xi_i} + \frac{\varphi_i}{\xi_i n(ct_i)^n}, \\ \Psi &= \Psi(t, t_i) = \frac{1}{ct}\left[\Lambda(ct) - \Lambda(ct_i) - \frac{1}{n(ct_i)^{n-1}}\right] \end{aligned} \tag{3.10}$$

In (3.10) the second expression for x differs from the first in terms that are quadratic in δ_i and φ_i.

When $n>0$, which corresponds to $\kappa>1$, the correction to the velocity of the particles at the stage of the expansion of the gas connected with $\partial p/\partial x \neq 0$, according to (3.10), remains small for all $t>t_i$, increasing monotonically (in modulus) from zero when $t=t_i$ to $|\varphi_i|/[n(ct_i)^n]$ as $t\to\infty$. Despite this, the trajectories of the particles may differ as much as we please not only from the rays $x=\xi_i ct$, but also from rectilinear trajectories of inertial motion of the particles (3.1). As can be seen from (3.10), this occurs if $n\leqslant 1$ or, by the definition of n, if $\kappa\leqslant\kappa_\nu = 1+1/(1+\nu)$. When $\kappa\leqslant\kappa_\nu$ in the formula for x from (3.10) the term proportional to $\Delta(ct)$, increases without limit as t increases, although more slowly than $\xi_i ct$. This result agrees with the conclusion reached in [1], with the sole difference that the analysis in [1] only holds in the region of the xt plane, extended along the t axis. It can be shown that at the boundary of the expanding gas with the vacuum $\varphi_i=0$. Hence, the limiting trajectory, like the trajectory of the particle, which is at rest when $x=0$, is rectilinear for any κ.

The Lagrangian form of writing the solution in the form (3.10) easily enables us to obtain the required formula for ρ. In fact, the connection between ρ at an arbitrary point of the trajectory and $\rho_i=\rho(x_i, t_i)$ can be written, by virtue of the equation of continuity, in the form [13]

$$\rho J=\rho_i,\quad J=\left(\frac{x}{x_i}\right)^\nu \frac{\partial x}{\partial x_i} \tag{3.11}$$

where J is the Jacobian of the transformation for the one-dimensional case considered. With the same accuracy as above we obtain from (3.10)

$$\frac{\partial x}{\partial x_i}=\beta_i\frac{t}{t_i}(1-\delta_i'\frac{t_i}{t}+\varphi_i'\Psi),\quad \beta_i=1+\delta_i'+\frac{\varphi_i'}{n(ct_i)^n} \tag{3.12}$$

Here, as earlier, the primes denote derivatives with respect to ξ_i,

Substituting (3.10) and (3.12) into (3.13) we obtain

$$\rho(\xi_i,t)=\frac{\rho_i}{\alpha_i^\nu\beta_i}\left(\frac{t_i}{t}\right)^{1+\nu}\left(1-\frac{\delta_i t_i}{\xi_i t}+\frac{\varphi_i}{\xi_i}\Psi\right)^{-\nu}\left(1-\delta_i'\frac{t_i}{t}+\varphi_i'\Psi\right)^{-1} \tag{3.13}$$

For small values of ξ_i, for which $\delta_i/\xi_i\simeq\delta_i'$, $\varphi_i/\xi_i\simeq\varphi_i'$ and $\beta_i\simeq\alpha$, (3.13) reduces to the corresponding formula from [1].

For $t\geqslant t_i$ along each trajectory defined by a fixed value of ξ_i, in any approximation the entropy function $\kappa(\xi_i)=p/\rho^\kappa$ is conserved. As a result of this $p(\xi_i, t)=k(\xi_i)\rho^\kappa$ with $\rho(\xi_i, t)$ from (3.13).

This approach holds provided all the corrections to the asymptotic formulae of the first approximation are small. If $n>0$, this requirement can always be satisfied by choosing t_i to be sufficiently large. According (3.10) it should be such that

$$|\delta_i|\ll\xi_i,\quad |\varphi_i|\ll n(ct_i)^n\xi_i \tag{3.14}$$

By satisfying (3.14) we ensure that the corresponding perturbations are relatively small, including the ratios (to $\xi_i ct$) which increase without limit for $1<\kappa<\kappa_\nu$ and the deviations of the trajectories from rays $x=\xi_i ct$.

4. If $n=0$, i.e. $\xi=1$, the second inequality in (3.14), which now cannot be satisfied because of the choice of t_i, gives $\varphi_i\equiv\varphi(\xi_i)\equiv 0$. This means that as $\kappa\to 1$ at the stage of expansion (if it is realized) the trajectories of the particles are straight lines not only in the first but also in the second approximation. Since, however, in this case u^m and c are infinite, while above we assumed c to be finite, then, although the same result formally follows from (3.4) and (3.7) as $c \to \infty$ a passage to the limit as $\kappa \to 1$ is fundamentally required. This can be done in several ways.

We will rewrite the equation $Tds = dh - \rho^{-1}dp$ for an ideal gas in the form

$$\frac{1}{\rho}dp = a^2\left(\frac{d\rho}{\rho} + \frac{1}{\kappa}ds\right) \tag{4.1}$$

and we will consider the problem of piston which, for $t > 0$, expands at a constant velocity c, which we will take as the scale of u. Later we will let c approach infinity. Since for a gas with $\kappa = 1$ the rate of expansion into a vacuum is infinite, for any $c \leq \infty$ the gas will not lag behind the piston, and for $c < \infty$ the density will be non-zero. Substituting (4.1) with $\kappa = 1$ into (3.3) without υ^2/x and using the same method of changing to dimensionless form as when obtaining (1.2), we have

$$\frac{du}{dt} + \frac{a^2}{c^2}\left(\frac{1}{\rho}\frac{\partial \rho}{\partial x} + \frac{\partial s}{\partial x}\right) = 0 \tag{4.2}$$

When $\kappa = 1$, a is a function of s only. Hence, after the shock waves die away (if these arise when $t \leq t_i$) a^2, together with s, is conserved in a particle, i.e. it does not decrease as it does when $\kappa > 1$. Despite this, when estimating "orders of magnitude" the term with a^2 when $t \geq t_i$ and large values of t becomes much less than $u(\partial u/\partial x)$. This justifies the procedure of the solution used above and formulae (2.1), (2.3), (3.1), (3.2) and (3.7). In (3.7) $\varphi(\xi)$ must now be understood to be

$$\varphi(\xi) = -\frac{a^2}{c^2}\left(\frac{R'}{R} + S'\right) \equiv -\frac{F(\xi)}{c^2} \tag{4.3}$$

where, in general, a like R and S, is a function of ξ. Proceeding further in the same way as in Section 3, we obtain the expressions

$$\begin{gathered} U(\xi_i, t) = \xi_i + \delta_i - \varphi_i \ln(t/t_i) \\ x(\xi_i, t) = x_i + (\xi_i + \delta_i - \varphi_i)c(t - t_i) + \varphi_i ct \ln(t/t_i) \end{gathered} \tag{4.4}$$

which, unlike when determining φ_i, are identical with the limit (3.10) as $n \to 0$.

These expressions do not give solutions of the piston problem considered since when $\xi_i = \xi_i^m$ the non-flow condition $U(\xi_i^m, t) = 1$, due to the last term in the first equation of (4.4) for $t > t_i$, is satisfied with an error of the order of φ_i. We will not, however, introduce the corresponding corrections into (4.4), since in the limit (4.4) of interest to us for fixed t and $c \to \infty$ these corrections, by (4.3), disappear together with $\varphi_i \to 0$.

Passing to the desired limit, we will make the natural assumption that the finite function $F_-(\xi) = \lim F(\xi)$ exists as $c \to \infty$ and $0 \leq \xi \leq 1$ and we will change from ξ and U to $\xi^\circ = x/(c^\circ t) = \xi c/c^\circ$ and $U^\circ = u/c^\circ = Uc/c^\circ$. Here, unlike when $c \to \infty$, the constant c° with the dimensions of velocity is finite. It is convenient to take as c° the value $a = a^m$ when $t = t_i$. If the parameters of the gas are constant and $u_0 \equiv 0$ when $t = 0$, then in the limit as $c \to \infty$, which clearly corresponds to shock-free expansion into a vacuum, $a^m = a_0$. Finally, as $c \to \infty$, instead of (4.4) and (3.13) we obtain

$$\begin{gathered} U^\circ(\xi_i^\circ, t) = \xi_i^\circ + \delta_i^\circ, \quad x(\xi_i^\circ, t) = x_i + (\xi_i^\circ + \delta_i^\circ)c^\circ(t - t_i) \\ \rho(\xi_i^\circ, t) = \frac{\rho_i}{\alpha_i^\nu \beta_i}\left(\frac{t_i}{t}\right)^{1+\nu}\left(1 - \frac{\delta_i^\circ t_i}{\alpha_i \xi_i^\circ t}\right)^{-\nu}\left(1 - \frac{\delta_i^{\circ\prime} t_i}{\beta_i t}\right)^{-1} \\ \alpha_i = 1 + \delta_i^\circ/\xi_i^\circ, \quad \beta_i = 1 + \delta_i^{\circ\prime} \end{gathered} \tag{4.5}$$

which confirms the conclusions reached purely formally at the beginning of this section.

As t_i increases the correction $\delta_i^\circ = U_i^\circ - \xi_i^\circ \to 0$ and the formula for ρ when $t \geq t_i$ becomes (1.4) with $j = 1 + \nu$. On the other hand, if we replace c by c° and $R(\xi)$ by $R^\circ(\xi^\circ)$ in expression (2.1) for ρ, we obtain

$$R^\circ(\xi^\circ) = \lim_{c\to\infty} R(\xi)\left(\frac{c^\circ}{c}\right)^{1+\nu} = 0 \tag{4.6}$$

and, consequently, $\rho \equiv 0$ when $t > t_i$. Although the stage of inertial expansion of the gas, for which (4.5) hold, assumes that ρ is small, the expression for ρ from (1.4) with $j = 1+\nu$ and (4.5) are more pithy than the identity $\rho \equiv 0$. As has already been pointed out, $R(\xi) \not\equiv 0$ and $P(\xi) \not\equiv 0$ are the result of the gas "remembering" the non-self-similar stage of expansion.

According to Section 2, when $\kappa = 1$ we can compile from x, t, m, E and the entropy function $p_0/\rho = a_0^2$ a unique independent dimensionless combination: $\xi^\circ = x/(a_0 t)$. Recalling the observation made above regarding $R(\xi)$ and $P(\xi)$, we will now use the fact that, for large t, when the initial volume is represented by a point, of the defining parameters the constants m, E and a_0 remain. Then, the problem of a piston with a constant piston velocity c and its limit as $c \to \infty$ is the problem of the expansion of a gas with $\kappa = 1$ into a vacuum and will also be self-similar.

In the problem of a piston, putting

$$u(x,t) = a_0 U^\circ(\xi^\circ), \quad \rho(x,t) = \frac{m}{(a_0 t)^{1+\nu}} R^\circ(\xi^\circ), \quad \xi^\circ = \frac{x}{a_0 t} \tag{4.7}$$

to determine $U^\circ(\xi^\circ)$ and $R^\circ(\xi^\circ)$ when $0 \le \xi^\circ \le c/a_0$ from the second equation of (1.1) and from (4.2) with $s \equiv \text{const}$ we obtain

$$(U^\circ - \xi^\circ)R^{\circ\prime} - (1 + \nu - U^{\circ\prime} - \nu U^\circ/\xi^\circ)R^\circ = 0 \tag{4.8}$$

$$(U^\circ - \xi^\circ)R^\circ U^{\circ\prime} - R^{\circ\prime} = 0 \tag{4.9}$$

Here, unlike the previous analysis, in the second equation we retain $\partial p/\partial x = a_0^2(\partial\rho/\partial x)$. By (4.7) we must solve system (4.8) with the conditions

$$U^\circ(0) = 0, \quad U^\circ(c/a_0) = c/a_0 \tag{4.10}$$

Substituting $R^{\circ\prime}/R^\circ$ from (4.9) into (4.8) and introducing the notation $W = U^\circ - \xi^\circ$ we arrive at the following equation for determining W

$$W' = \frac{\xi^\circ W^2 - \nu W}{\xi^\circ(1 - W^2)} \tag{4.11}$$

the solution of which, by virtue of (4.10), must satisfy the conditions $W(0) = W(c/a_0) = 0$. A unique solution of Eq. (4.11) which satisfies these conditions is $W \equiv 0$, i.e. $U^\circ(\xi^\circ) = \xi^\circ$. Hence we also have from (4.9) $R^{\circ\prime} = 0$, and, consequently, $R^\circ(\xi) \equiv R^\circ(0) = R_0^\circ$. We obtain the constant R_0 from the rewritten integral condition of conservation of mass, taking (4.7) into account, namely, the first equation of (2.4) $R_0^\circ = (1+\nu)\varepsilon^{1+\nu}$, where $\varepsilon = a_0/c$. Hence in the limit as $c \to \infty$, which corresponds to the expansion of a gas with $\kappa = 1$, into a vacuum, we will have

$$U^\circ \equiv \xi^\circ, \quad R^\circ \equiv 0, \quad 0 \le \xi^\circ \le \infty \tag{4.12}$$

The second integral condition (2.4) in the problem of a piston is not satisfied, since the gas does work and its energy is not conserved. If this condition is rewritten taking into account the method used to change to dimensionless quantities and the fact that in the case investigated $h = h_0 + a_0^2 \ln(\rho/\rho_0)$, we obtain from it for the work A done by the gas

$$A = m a_0^2\left[\ln\frac{\rho_0}{\rho} - \frac{1+\nu}{2(3+\nu)}\left(\frac{c}{a_0}\right)^2\right]$$

This formula can be used beginning from those values of t, and, consequently, ρ also, for which the flow is fairly close to that for the self-similar solution with $R^\circ(\xi)\equiv\text{const}\neq 0$.

In deriving (4.9) we assumed that $s\equiv s_0\equiv\text{const}$.This is certainly true in the limiting case when $c=\infty$. Hence, applying the solution obtained above with $R^\circ(\xi)=\text{const}$ directly to this case and taking into account that on the "boundary" (when $\xi^\circ=\infty$) the density vanishes, we immediately obtain (4.12). It is true that with this approach the result obtained can be shown to be strange, as the boundary $\partial\Omega$, transfers instantaneously to infinity. But a gas with $\kappa=1$ really does.

As we have already noted, the relation between $R^\circ(\xi^\circ)$ and $R(\xi)$ as $c\to\infty$ is given by (4.6), which shows that the corresponding representations are not contradictory. As regards the problem of a piston with $c<\infty$, the self-similar solution (4.7) with $R^\circ(\xi)^\circ\equiv(1+\nu)\varepsilon^{1+\nu}$, which "forgets" the initial non-self-similar stage, is less accurate and pithy than the self-similar solution which also describes the dispersion of a gas as regards the rate of solution of the "first approximation" (2.1) and the non-self-similar solution of the "second approximation" (4.5).

5. The approach used in Section 3, enables us to take into account the effect of gravitation on the spherically symmetrical dispersion of a gas. The equation of motion in this case has the form [11, 14]

$$\frac{du}{dt}+\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{4\pi G}{x^2}\int_0^x \rho x^2 dx=0 \tag{5.1}$$

where G is the gravitational constant.

If, despite the presence of gravitation, the gas expands, then, in the first approximation, the third term in (5.1) can be neglected, and we then proceed in the same way as in Section 3. As a result, instead of (3.10) and (3.13) we obtain

$$U(\xi_i,t)=\xi_i+\delta_i+\frac{\varphi_i}{n}\left[\frac{1}{(ct_i)^n}-\frac{1}{(ct)^n}\right]+\frac{f_i}{ct}\ln\frac{t}{t_i} \tag{5.2}$$

$$x(\xi_i,t)=x_i+\left[\xi_i+\delta_i+\frac{\varphi_i}{n(ct_i)^n}+\frac{f_i}{ct_i}\right]c(t-t_i)+\varphi_i[\Lambda(ct)-\Lambda(ct_i)]-$$

$$-f_i\ln\frac{t}{t_i}\simeq x_i\alpha_i\frac{t}{t_i}\left[1-\frac{\delta_i t_i}{\xi_i t}+\frac{\varphi_i}{\xi_i}\Psi-\frac{f_i}{\alpha_i\xi_i ct}\left(1+\ln\frac{t}{t_i}\right)\right]$$

$$\rho(\xi_i,t)=\frac{\rho_i}{\alpha_i^\nu\beta_i}\left(\frac{t_i}{t}\right)^{1+\nu}\left[1-\frac{\delta_i t_i}{\xi_i t}+\frac{\varphi_i}{\xi_i}\Psi-\frac{f_i}{\alpha_i\xi_i ct}\left(1+\ln\frac{t}{t_i}\right)\right]^{-\nu}\times$$

$$\times\left[1-\delta_i'\frac{t_i}{t}+\varphi_i'\Psi-\frac{f_i'}{\beta_i ct}\left(1+\ln\frac{t}{t_i}\right)\right]^{-1}$$

$$f(\xi)=-\frac{4\pi mG}{c^2\xi^2}\int_0^\xi R(\xi)\xi^2 d\xi$$

$$\alpha_i=1+\frac{\delta_i}{\xi_i}+\frac{\varphi_i}{\xi_i n(ct_i)^n}+\frac{f_i}{\xi_i ct_i},\quad \beta_i=1+\delta_i'+\frac{\varphi_i'}{n(ct_i)^n}+\frac{f_i'}{ct_i}$$

Here δ, φ, ∇ and Ψ are the same as in (3.10), $f_i=f(\xi_i)$, while the primes, as before, denote derivatives with respect to ξ_i. The formulae of the "second approximation", which describe the dispersion of a gravitating gas with $\kappa=1$, can be obtained from (5.2) with $c\to\infty$ and do not differ from (4.5). The question of whether one can obtain almost inertial expansion in a gravitating gas is determined by the evolution of the flow at its initial stage [11, 14].

I wish to thank A. L. Ni for useful discussions.
This research was supported by the Russian Fund for Fundamental Research (93-013-17514).

REFERENCES

1. KRAIKO A. N., The free unsteady expansion of an ideal gas. *Izv. Ross. Akad. Nauk. MZhG.* 4, 155–163, 1993.
2. LADYZHENSKII M. D., Some properties of hypersonic flows. *Trudy TsAGI* 778, 1960.
3. LADYZHENSKII M. D., Gas flows at high supersonic velocity. *Dokl. Akad. NaukSSSR* **134**, 2, 296–299, 1960.
4. GUSEV V. N. and LADYZHENSKII M. D., A gas-dynamic calculation of shock tubes and hypersonic nozzles under conditions of equilibrium dissociation and ionization of air. *Trudy TsAGI* 779, 1960.
5. LADYZHENSKII M. D., Analysis of the equations of hypersonic flows and the solution of the Cauchy problem. *Prikl. Mat. Mekh.* **26**, 2, 289–299, 1962.
6. LADYZHENSKII M. D., Hypersonic flows in nozzles. *Prikl. Mat. Mekh.* **29**, 1, 99–105, 1965.
7. LADYZHENSKII M. D., *Spatial Hypersonic Gas Flows.* Mashinostroyeniye, Moscow, 1968.
8. KRAIKO A. N. and SHELOMOVSKII V. V., Free expansion of two-dimensional jets of an ideal gas. *Prikl. Mat. Mekh.* **44**, 2, 271–280, 1980.
9. LIFSHITS Yu. B. and MAREVTSEVA N. A., The free expansion of three-dimensional jets of an ideal gas. *Prikl. Mat. Mekh.* **47**, 3, 428–432, 1983.
10. CHERNYI G. G., *Gas Dynamics*. Nauka, Moscow, 1988.
11. SEDOV L. I., *The Methods of Similitude and Dimensions in Mechanics*. Nauka, Moscow, 1972.
12. BARENBLATT G. I., *Similitude, Self-similarity and Intermediate Asymptotic Form. Theory and Applications to Geophysical Hydrodynamics*. Gidrometeoizdat, Leningrad, 1977.
13. ZEL'DOVICH Ya. B. and MYSHKIS A. D., *The Elements of Mathematical Physics.* Nauka, Moscow, 1973.
14. BOGOYAVLENSKII O. I., *Methods of the Qualitative Theory of Dynamic Systems in Astrophysics and Gas Dynamics.* Nauka, Moscow, 1980.

Translated by R.C.G.